The syntactic graph of a sofic shift

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Abstract. We define a new invariant for the conjugacy of irreducible sofic shifts. This invariant, that we call the syntactic graph of a sofic shift, is the directed acyclic graph of characteristic groups of the non null regular $\mathcal{D}$-classes of the syntactic semigroup of the shift.

Keywords: Automata and formal languages, symbolic dynamics.

1 Introduction

Sofic shifts [17] are sets of bi-infinite labels in a labeled graph. If the graph can be chosen strongly connected, the sofic shift is said to be irreducible. A particular subclass of sofic shifts is the class of shifts of finite type, defined by a finite set of forbidden blocks. Two sofic shifts X and Y are conjugate if there is a bijective block map from X onto Y. It is an open question to decide whether two sofic shifts are conjugate, even in the particular case of irreducible shifts of finite type.

There are many invariants for conjugacy of subshifts, algebraic or combinatorial, see [13, Chapter 7], [6], [12], [3]. For instance the entropy is a combinatorial invariant which gives the complexity of allowed blocks in a shift. The zeta function is another invariant which counts the number of periodic orbits in a shift.

In this paper, we define a new invariant for irreducible sofic shifts. This invariant is based on the structure of the syntactic semigroup of the language of finite blocks of the shift. Irreducible sofic shifts have a unique (up to isomorphisms of automata) minimal deterministic presentation, called the right Fischer cover of the shift. The syntactic semigroup S of an irreducible sofic shift is the transition semigroup of its right Fischer cover.

In general, the structure of a finite semigroup is determined by the Green's relations (denoted $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}, \mathcal{J}$) [16]. Our invariant is the acyclic directed graph whose nodes are the characteristic groups of the non null regular $\mathcal{D}$-classes of S. The edges correspond to the partial order $\leq_{\mathcal{J}}$ between these $\mathcal{D}$-classes. We call it the syntactic graph of the sofic shift. The result can be extended to the case of reducible sofic shifts.

The proof of the invariant is based on Nasu's Classification Theorem for sofic shifts [15] that extends William's one for shifts of finite type. This theorem says that two irreducible sofic shifts X, Y are conjugate if and only if there is a sequence of transition matrices of right Fischer covers $A = A_0, A_1, \ldots, A_{l-1}, A_l = B$, such that A_{i-1}, A_i are elementary strong shift equivalent for $1 \leq i \leq l$, where

A and B are the transition matrices of the right Fischer covers of X and Y, respectively. This means that there are transition matrices U_i, V_i such that, after recoding the alphabets of A_{i-1} and A_i, we have $A_{i-1} = U_iV_i$ and $A_i = V_iU_i$. A bipartite shift is associated in a natural way to a pair of elementary strong shift equivalent and irreducible sofic shifts [15].

The key point in our invariant is the fact that an elementary strong shift equivalence relation between transition matrices implies some conjugacy relations between the idempotents in the syntactic semigroup of the bipartite shift.

We show that particular classes of irreducible sofic shifts can be characterized with this syntactic invariant: the class of irreducible shifts of finite type and the class of irreducible aperiodic sofic shifts.

Basic definitions related to symbolic dynamics are given in Section 2.1. We refer to [13] or [9] for more details. See also [10], [11], [4] about sofic shifts. Basic definitions and properties related to finite semigroups and their structure are given Section 2.2. We refer to [16, Chapter 3] for a more comprehensive expository. Nasu's Classification Theorem is recalled in Section 2.3. We define and prove our invariant in Section 3. A comparison of this syntactic invariant to some well known other ones is given in Section 4. Proofs of Propositions 1 and 2 are omitted. The extension to the case of reducible sofic shifts is discussed at the end of Section 3.

2 Definitions and background

2.1 Sofic shifts and their presentations

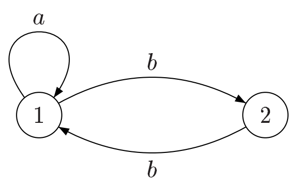
Let $\mathcal{A}$ be a finite alphabet, i.e. a finite set of symbols. The shift map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$, for $(a_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$. If $\mathcal{A}^{\mathbb{Z}}$ is endowed with the product topology of the discrete topology on $\mathcal{A}$, a *subshift* is a closed σ-invariant subset of $\mathcal{A}^{\mathbb{Z}}$.

If X is a subshift of $\mathcal{A}^{\mathbb{Z}}$ and n a positive integer, the nth *higher power* of X is the subshift of $(\mathcal{A}^n)^{\mathbb{Z}}$ defined by $X^n = \{(a_{in}, \ldots, a_{in+n-1})_{i\in\mathbb{Z}} \mid (a_i)_{i\in\mathbb{Z}} \in X\}$.

A finite *automaton* is a finite multigraph labeled on $\mathcal{A}$. It is denoted $\mathsf{A} = (Q, E)$, where Q is a finite set of states, and E a finite set of edges labeled on $\mathcal{A}$. It is equivalent to a *symbolic adjacency* $(Q \times Q)$*-matrix* A, where A_{pq} is the finite formal sum of the labels of all the edges from p to q. A *sofic shift* is the set of the labels of all the bi-infinite paths on a finite automaton. If A is a finite automaton, we denote by X_A the sofic shift defined by the automaton A. Several automata can define the same sofic shift. They are also called *presentations* or *covers* of the sofic shift. We will assume that all presentations are *essential*: all states have at least one outgoing edge and one incoming edge. An automaton is *deterministic* if for any given state and any given symbol, there is at most one outgoing edge labeled with this given symbol. A sofic shift is *irreducible* if it has a presentation with a strongly connected graph. Irreducible sofic shifts have a unique (up to isomorphisms of automata) minimal deterministic presentation called the *right Fischer cover* of the shift.

Let $\mathsf{A} = (Q, E)$ be a finite deterministic (essential) automaton on the alphabet $\mathcal{A}$. Each finite word w of $\mathcal{A}^*$ defines a partial function from Q to Q. This function sends the state p to the state q, if w is the label of a path form p to q. The semigroup generated by all these functions is called the *transition semigroup* of the automaton. When X_A is not the full shift, the semigroup has a null element, denoted 0, which corresponds to words which are not factors of any bi-infinite word of X_A. The *syntactic semigroup* of an irreducible sofic shift is defined as the transition semigroup of its right Fischer cover.

Example 1. The sofic shift presented by the automaton of Figure 1 is called the *even shift*. Its syntactic semigroup is defined by the table in the right part of the figure.



	1	2
a	1	−
b	2	1
ab	2	−
ba	−	1
bb	1	2
bab	−	2
aba	−	−

.

Fig. 1. The right Fischer cover of the even shift and its syntactic semigroup. Since aa and a define the same partial function from Q to Q, we write $aa = a$ in the syntactic semigroup. We also have $aba = 0$, or $ab^{2k+1}a = 0$ for any nonnegative integer k. The word bb is the identity in this semigroup.

2.2 Structure of finite semigroups

We refer to [16] for more details about the notions defined in this section.

Given a semigroup S, we denote by S^1 the following monoid: if S is a monoid, $S^1 = S$. If S is not a monoid, $S^1 = S \cup \{1\}$ together with the law $*$ defined by $x * y = xy$ if $x, y \in S$ and $1 * x = x * 1 = x$ for each $x \in S^1$.

We recall the *Green's relations* which are fundamentals equivalence relations defined in a semigroup S. The four equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{J}$ are defined as follows. Let $x, y \in S$,

$$x \mathcal{R} y \Leftrightarrow xS^1 = yS^1,$$
$$x \mathcal{L} y \Leftrightarrow S^1 x = S^1 y,$$
$$x \mathcal{J} y \Leftrightarrow S^1 x S^1 = S^1 y S^1,$$
$$x \mathcal{H} y \Leftrightarrow x \mathcal{R} y \text{ and } x \mathcal{L} y.$$

Another relation $\mathcal{D}$ is defined by:

$$x\mathcal{D}y \Leftrightarrow \exists z \in S \;\; x\mathcal{R}z \text{ and } z\mathcal{L}y.$$

In a finite semigroup $\mathcal{J} = \mathcal{D}$. We recall the definition of the quasi-order $\leq_{\mathcal{J}}$:

$$x \leq_{\mathcal{J}} y \Leftrightarrow S^1xS^1 \subseteq S^1yS^1.$$

An $\mathcal{R}$-*class* is an equivalence class for a relation $\mathcal{R}$ (similar notations hold for the other Green's relations). An *idempotent* is an element $e \in S$ such that $ee = e$. A *regular* class is a class containing an idempotent. In a regular $\mathcal{D}$-class, any $\mathcal{H}$-class containing an idempotent is a maximal subgroup of the semigroup. Moreover, two regular $\mathcal{H}$-classes contained in a same $\mathcal{D}$-class are isomorphic (as groups), see for instance [16, Proposition 1.8]. This group is called the *characteristic group* of the regular $\mathcal{D}$-class. The quasi-order $\leq_{\mathcal{J}}$ induces a partial order between the $\mathcal{D}$-classes (still denoted $\leq_{\mathcal{J}}$). The structure of the transition semigroup S is often described by the so called "egg-box" pictures of the $\mathcal{D}$-classes.

We say that two elements $x, y \in S$ are conjugate if there are elements $u, v \in S^1$ such that $x = uv$ and $y = vu$. Two idempotents belong to a same regular $\mathcal{D}$-class if and only if they are conjugate, see for instance [16, Proposition 1.12].

Let S be a transition semigroup of an automaton $\mathsf{A} = (Q, E)$ and $x \in S$. The *rank* of x is the cardinal of the image of x as a partial function from Q to Q. The *kernel* of x is the partition induced by the equivalence relation $\sim$ over the domain of x where $p \sim q$ if and only p, q have the same image by x. The kernel of x is thus a partition of the domain of x. We describe the egg-box pictures with Example 1 continued in Figure 2.

$$\begin{array}{r|c|} & 12 \\ \hline 1/2 & \begin{array}{c} b \\ {}^*b^2 \end{array} \\ \hline \end{array} \qquad \begin{array}{r|c|c|} & 1 & 2 \\ \hline 1 & {}^*a & ab \\ \hline 2 & ba & {}^*bab \\ \hline \end{array} \qquad \begin{array}{r|c|} & - \\ \hline - & {}^*0 \\ \hline \end{array}$$

Fig. 2. The syntactic semigroup of the even shift of Example 1 is composed of three $\mathcal{D}$-classes D_1, D_2, D_3, of rank 2, 1 and 0, respectively, represented by the above tables from left to right. Each square in a table represents an $\mathcal{H}$-class. Each row represents an $\mathcal{R}$-class and each column an $\mathcal{L}$-class. The common kernel of the elements in each row is written on the left of each row. The common image of the elements in each column is written above each column. Idempotents are marked with the symbol $*$. Each $\mathcal{D}$-class of this semigroup is regular. The characteristic groups of D_1, D_2, D_3 are $\mathbb{Z}/2\mathbb{Z}$, the trivial group $\mathbb{Z}/\mathbb{Z}$ and $\mathbb{Z}/\mathbb{Z}$, respectively.

Let X be an irreducible sofic shift and S its syntactic semigroup. It is known that S has a unique $\mathcal{D}$-class of rank 1 which is regular (see [4] or [5], see also [8]).

We define a finite directed acyclic graph (DAG) associated with X as follows. The set of vertices of the DAG is the set of non null regular $\mathcal{D}$-classes of S, but the regular $\mathcal{D}$-class of null rank, if there is one. Each vertex is labeled with the rank of the $\mathcal{D}$-class and its characteristic group. There is an edge from the vertex associated with a $\mathcal{D}$-class D to the vertex associated with a $\mathcal{D}$-class D' if and only if $D' \leq_{\mathcal{J}} D$. We call this acyclic graph the *syntactic graph* of X (see Figure 3 for an example). Note that the regular $\mathcal{D}$-class of null rank, if there is one, is not taken into account in a syntactic graph. This is linked to the fact that a full shift (i.e. the set of all bi-infinite words on a finite alphabet) can be conjugate to a non full shift.

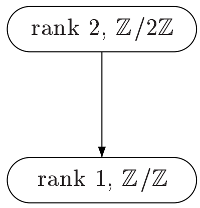


Fig. 3. The syntactic graph of the even shift of Example 1. We have $D_2 \leq_{\mathcal{J}} D_1$ since, for instance, $S^1abS^1 \subseteq S^1bS^1$.

2.3 Nasu's Classification Theorem for sofic shifts

In this section, we recall Nasu's Classification Theorem for sofic shifts [15] (see also [13, p. 232]), which extends William's Classification Theorem for shifts of finite type (see [13, p. 229]).

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}, Y \subseteq \mathcal{B}^{\mathbb{Z}}$ be two subshifts and m, a be nonnegative integers. A map $\phi : X \to Y$ is a (m, a)-*block map* (or (m, a)-*factor map*) if there is a map $\delta : \mathcal{A}^{m+a+1} \to \mathcal{B}$ such that $\phi((a_i)_{i\in\mathbb{Z}}) = (b_i)_{i\in\mathbb{Z}}$ where $\delta(a_{i-m} \dots a_{i-1}a_ia_{i+1} \dots a_{i+a}) = b_i$. A *block map* is a (m, a)-*block map* for some nonnegative integers m, a. The well known theorem of Curtis, Hedlund and Lyndon [7] asserts that continuous and shift-commuting maps are exactly block maps. A *conjugacy* is a one-to-one and onto block map (then, being a shift compact, its inverse is also a block map).

Let A be a symbolic adjacency $(Q \times Q)$-matrix of an automaton A with entries in a finite alphabet $\mathcal{A}$. Let $\mathcal{B}$ be a finite alphabet and f a one-to-one map from $\mathcal{A}$ to $\mathcal{B}$. The map f is extended to a morphism from finite formal sums of elements of $\mathcal{A}$ to finite formal sums of elements of $\mathcal{B}$. We say that f *transforms* A into an adjacency $(Q \times Q)$-matrix B if $B_{pq} = f(A_{pq})$.

We now define the notion of strong shift equivalence between two symbolic adjacency matrices.

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite alphabets. We denote by $\mathcal{AB}$ the set of words ab with $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Two symbolic adjacency matrices A, with entries in $\mathcal{A}$, and B, with entries in $\mathcal{B}$, are *elementary strong shift equivalent* if there is a pair of symbolic adjacency matrices (U, V) with entries in disjoint alphabets $\mathcal{U}$ and $\mathcal{V}$ respectively, such that there is a one-to-one map from $\mathcal{A}$ to $\mathcal{UV}$ which transforms A into UV, and there is a one-to-one map from $\mathcal{B}$ to $\mathcal{VU}$ which transforms B into VU.

Two symbolic adjacency matrices A and B are *strong shift equivalent within right Fischer covers* if there is a sequence of symbolic adjacency matrices of right Fischer covers

$$A = A_0, A_1, \ldots, A_{l-1}, A_l = B$$

such that for $1 \leq i \leq l$ the matrices A_{i-1} tand A_i are elementary strong shift equivalent.

Theorem 1 (Nasu). *Let X and Y be irreducible sofic shifts and let A and B be the symbolic adjacency matrices of the right Fischer covers of X and Y, respectively. Then X and Y are conjugate if and only if A and B are strong shift equivalent within right Fischer covers.*

Example 2. Let us consider the two (conjugate) irreducible sofic shifts X and Y defined by the right Fischer covers $\mathsf{A} = (Q, E)$ and $\mathsf{B} = (Q', E')$ in Figure 4.

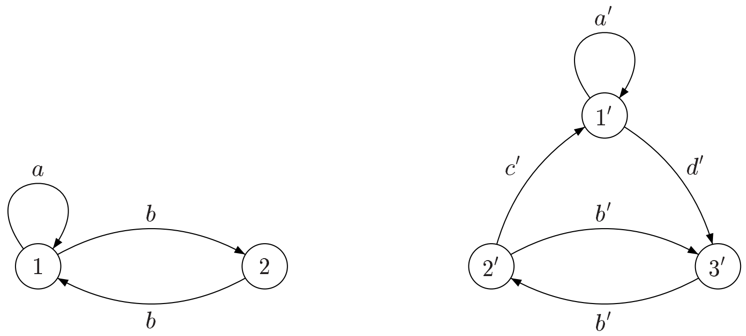


Fig. 4. Two conjugate shifts X and Y.

The symbolic adjacency matrices of these automata are respectively

$$A = \begin{bmatrix} a & b \\ b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} a' & 0 & d' \\ c' & 0 & b' \\ 0 & b' & 0 \end{bmatrix}.$$

Then A and B are elementary strong shift equivalent with

$$U = \begin{bmatrix} u_1 & 0 & u_2 \\ 0 & u_2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & 0 \\ v_2 & 0 \\ 0 & v_2 \end{bmatrix}.$$

Indeed,

$$UV = \begin{bmatrix} u_1v_1 & u_2v_2 \\ u_2v_2 & 0 \end{bmatrix}, \quad VU = \begin{bmatrix} v_1u_1 & 0 & v_1u_2 \\ v_2u_1 & 0 & v_2u_2 \\ 0 & v_2u_2 & 0 \end{bmatrix}.$$

The one-to-one maps from $\mathcal{A} = \{a, b\}$ to $\mathcal{UV}$ and from $\mathcal{B} = \{a', b', c', d'\}$ to $\mathcal{VU}$ are described in the tables below.

a	u_1v_1
b	u_2v_2

,

a'	v_1u_1
b'	v_2u_2
c'	v_2u_1
d'	v_1u_2

.

An elementary strong shift equivalence enables the construction of an irreducible sofic shift Z on the alphabet $\mathcal{U} \cup \mathcal{V}$ as follows. The sofic shift Z is defined by the automaton $\mathsf{C} = (Q \cup Q', F)$, where the symbolic adjacency matrix C of C is

$$\begin{array}{cc} & \begin{array}{cc} Q & Q' \end{array} \\ \begin{array}{c} Q \\ Q' \end{array} & \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix} \end{array}.$$

The shift Z is called the *bipartite shift* defined by U, V (see Figure 5). An edge of C labeled on $\mathcal{U}$ goes from a state in Q to a state in Q'. An edge of C labeled on $\mathcal{V}$ goes from a state in Q' to a state in Q. Remark that the second higher power of Z is the disjoint union of X and Y. Note also that C is a right Fischer cover (i.e. is minimal).

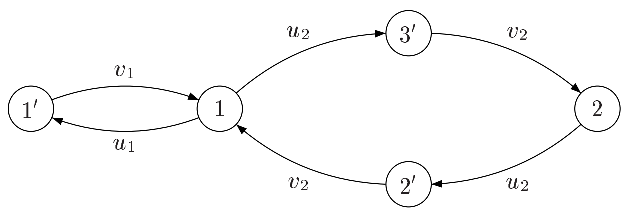


Fig. 5. The bipartite shift Z.

3 A syntactic invariant

In this section, we define a syntactic invariant for the conjugacy of irreducible sofic shifts.

Theorem 2. *Let X and Y be two irreducible sofic shifts. If X and Y are conjugate, then they have the same syntactic graph.*

We give a few lemmas before proving Theorem 2.

Let X (respectively Y) be an irreducible sofic shift whose symbolic adjacency matrix of its right Fischer cover is a $(Q \times Q)$-matrix (respectively $(Q' \times Q')$-matrix) denoted by A (respectively by B). We assume that A and B are elementary strong shift equivalent through a pair of matrices (U, V). The corresponding alphabets are denoted $\mathcal{A}$, $\mathcal{B}$, $\mathcal{U}$, and $\mathcal{V}$ as before. We denote by f a one-to-one map from $\mathcal{A}$ to $\mathcal{U}\mathcal{V}$ which transforms A into UV and by g a one-to-one map from $\mathcal{B}$ to $\mathcal{V}\mathcal{U}$ which transforms B into VU. Let Z be the bipartite irreducible sofic shift associated to U, V. We denote by S (respectively T, R) the syntactic semigroup of X (respectively Y, Z).

Let $w \in R$. If w is non null, the bipartite nature of Z implies that w is a function from $Q \cup Q'$ to $Q \cup Q'$ whose domain is included either in Q or in Q', and whose image is included either in Q or in Q'. If $w \neq 0$ with a domain included in P and an image included in P', we say that w has the *type* (P, P'). Remark that w has type (Q, Q) if and only if $w \neq 0$ and $w \in (f(\mathcal{A}))^*$, and w has type (Q', Q') if and only if $w \neq 0$ and $w \in (g(\mathcal{B}))^*$.

Lemma 1. *Elements of R in a same non null $\mathcal{H}$-class have the same type.*

Proof We show the property for the (Q, Q)-type. Let $w \in H$ and w of type (Q, Q). If $w = w'v$ with $w', v \in R$, then w' has type $(Q, *)$. If $w = zw'$ with $z, w' \in R$, then w' has type $(*, Q)$. Thus, $w\mathcal{H}w'$ implies that w' has type (Q, Q). □

The $\mathcal{H}$-classes of R containing elements of type (Q, Q) (respectively (Q', Q')) are called (Q, Q)-$\mathcal{H}$-*classes* (respectively (Q', Q')-$\mathcal{H}$-classes).

Let $w = a_1 \ldots a_n$ be an element of S, we define the element $f(w)$ as $f(a_1) \ldots f(a_n)$. Note that this definition is consistent since if $a_1 \ldots a_n = a'_1 \ldots a'_m$ in S, then $f(a_1) \ldots f(a_n) = f(a'_1) \ldots f(a'_m)$ in R. Similarly we define an element $g(w)$ for any element w of T.

Conversely, let w be an element of R belonging to $f(\mathcal{A})^*$ ($\subseteq (\mathcal{U}\mathcal{V})^*$). Then $w = f(a_1) \ldots f(a_n)$, with $a_i \in \mathcal{A}$. We define $f^{-1}(w)$ as $a_1 \ldots a_n$. Similarly we define $g^{-1}(w)$. Again these definitions and notations are consistent. Thus f is a semigroup isomorphism from S to the subsemigroup of R of transition functions defined by the words in $(f(\mathcal{A}))^*$. Notice that $f(0) = 0$ if $0 \in S$. Analogously, g is a semigroup isomorphism from T to the subsemigroup of R of transition functions defined by the words in $(g(\mathcal{B}))^*$.

Lemma 2. *Let $w, w' \in R$ of type (Q, Q). Then $w\mathcal{H}w'$ in R if and only if $f^{-1}(w)\mathcal{H}f^{-1}(w')$ in S.*

Proof Let $w = f(a_1) \ldots f(a_n)$ and $w' = f(a'_1) \ldots f(a'_m)$, with $a_i, a'_j \in \mathcal{A}$. We have $w = w'v$ with $v \in R$ if and only if $v = f(\bar{a}_1) \ldots f(\bar{a}_r)$ with $\bar{a}_i \in \mathcal{A}$ and $f(a_1) \ldots f(a_n) = f(a'_1) \ldots f(a'_m) f(\bar{a}_1) \ldots f(\bar{a}_r)$. This is equivalent to $a_1 \ldots a_n = a'_1 \ldots a'_m \bar{a}_1 \ldots \bar{a}_r$, that is $f^{-1}(w)R^1 \subseteq f^{-1}(w')R^1$. Analogously, we have $w' = wv'$ with $v' \in R$, if and only if $f^{-1}(w')R^1 \subseteq f^{-1}(w)R^1$. This proves that $w\mathcal{R}w'$

in R if and only if $f^{-1}(w)\mathcal{R}f^{-1}(w')$ in S. In the same way, one can prove the same statement for the relation $\mathcal{L}$ and hence for the relation $\mathcal{H}$. $\square$

A similar statement holds for (Q', Q')-$\mathcal{H}$-classes.

Lemma 3. *Let $w, w' \in R$ of type (Q, Q). Then $w \leq_{\mathcal{J}} w'$ in R if and only if $f^{-1}(w) \leq_{\mathcal{J}} f^{-1}(w')$ in S. This implies that $w\mathcal{J}w'$ in R if and only if $f^{-1}(w)$ $\mathcal{J}$ $f^{-1}(w')$ in S.*

Proof The first statement can be prooved as in the previous lemma. $\square$

Similar results hold between T and R. As a consequence we get the following lemma.

Lemma 4. *The bijection f between S and the elements of R belonging to $(f(\mathcal{A}))^*$, induces a bijection between the non null $\mathcal{H}$-classes of S and the (Q, Q)-$\mathcal{H}$-classes of R. Moreover this bijection keeps the relations $\mathcal{J}$, $\leq_{\mathcal{J}}$ and the rank of the $\mathcal{H}$-classes.*

A similar statement holds for the bijection g.

We now come to the main lemma, which shows the link between the elementary strong shift equivalence of the symbolic adjacency matrices and the conjugacy of some idempotents in the semigroup. This link is the key point of the invariant.

Lemma 5. *Let H be a regular (Q, Q)-$\mathcal{H}$-class of R. Then there is a regular (Q', Q')-$\mathcal{H}$-class in the same $\mathcal{D}$-class as H.*

Proof Let $e \in R$ be an idempotent element of type (Q, Q). Let $u_1v_1 \dots u_nv_n$ in $(\mathcal{U}\mathcal{V})^*$ such that $e = u_1v_1 \dots u_nv_n$. We define $\bar{e} = v_1 \dots u_nv_nu_1$. Thus $eu_1 = u_1\bar{e}$ in R. Remark that $\bar{e}$ depends on the choice of the word $u_1v_1 \dots u_nv_n$ representing e in R.

If w denotes $v_1 \dots u_nv_n$ and v denotes u_1, we have $e = vw$ and $\bar{e} = wv$. It follows that e and $\bar{e}$ are conjugate, thus $e^2 = e$ and $\bar{e}^2$ are conjugate. Moreover

$$\bar{e}^3 = wvwvwv = weev = wev = wvwv = \bar{e}^2.$$

Thus $\bar{e}^2$ is an idempotent conjugate to the idempotent e. As a consequence e and $\bar{e}^2$ belong to a same $\mathcal{D}$-class of R (see Section 2), and $\bar{e}^2 \neq 0$. The result follows since $\bar{e}^2$ is of type (Q', Q'). $\square$

Note that the number of regular (Q, Q)-$\mathcal{H}$-classes and the number of regular (Q', Q')-$\mathcal{H}$-classes in a same $\mathcal{D}$-class of R, may be different in general.

We now prove Theorem 2.

Proof[of Theorem 2] By Nasu's Theorem [15] we can assume, without loss of generality, that the symbolic adjacency matrices of the right Fischer covers of X and Y are elementary strong shift equivalent. We define the bipartite shift Z

as above. We denote by S, T and R the syntactic semigroups of X, Y and Z respectively.

Let D be a non null regular $\mathcal{D}$-class of S. Let H be a regular $\mathcal{H}$-class of S contained in D. Let $H'' = f(H)$. By Lemma 4, the groups H and H'' are isomorphic. Let D'' the $\mathcal{D}$-class of R containing H''. By Lemma 5, there is at least one regular (Q', Q')-$\mathcal{H}$-class K'' in D'', which is isomorphic to H''. Let $H' = g^{-1}(K'')$ and let D' be the $\mathcal{D}$-class of T containing H'. By Lemma 4, the groups H' and K'' are isomorphic. Hence the groups H and H' are isomorphic.

By Lemmas 4 and 5, we have that the above construction of D' from D is a bijective function φ from the non null regular $\mathcal{D}$-classes of S onto the non null regular $\mathcal{D}$-classes of T. Moreover the characteristic group of D is isomorphic to the characteristic group of $\varphi(D)$ and, by Lemma 4, the rank of D is equal to the rank of $\varphi(D)$.

We now consider two non null regular $\mathcal{D}$-classes D_1 and D_2 of S. By Lemma 4 and Lemma 5, $D_1 \leq_{\mathcal{J}} D_2$ if and only if $\varphi(D_1) \leq_{\mathcal{J}} \varphi(D_2)$. It follows that the syntactic graphs of S and T are isomorphic through the bijection φ. $\square$

Nasu's Classification Theorem holds for reducible sofic shifts by the use of right Krieger covers instead of right Fischer covers [15]. This enables the extension of our result to the case of reducible sofic shifts. This extension is not described in this short version of the paper.

4 How dynamic is this invariant?

We briefly compare the syntactic conjugacy invariant with other classical conjugacy invariants. We refer to [13] for the definitions and properties of these classical invariants.

First, on can remark that the syntactic invariant does not capture all the dynamic. Two sofic shifts can have the same syntactic graph and a different entropy, see the example given in Figure 6.

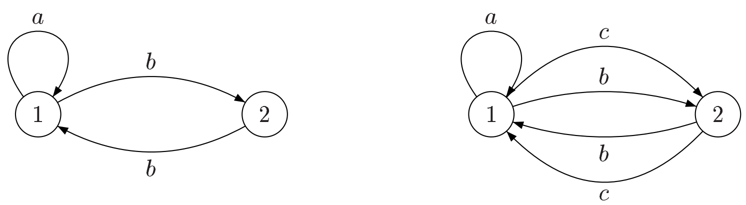


Fig. 6. The two above sofic shifts X, Y have the same syntactic graph and a different entropy. Indeed, we have $b = c$ in the syntactic semigroup of Y. Hence the shifts X and Y have the same syntactic semigroup.

The comparison with the zeta function is more interesting. Recall that the zeta function of a shift X is $\zeta(X) = \exp \sum_{n \geq 1} p_n \frac{z^n}{n}$, where p_n is the number of bi-infinite words $x \in X$ such that $\sigma^n(x) = x$. We give in Figure 7 an example of two irreducible sofic shifts which have the same zeta function and different syntactic graphs.

Irreducible shifts of finite type can be characterized with this syntactic invariant. Other equivalent characterizations of finite type shifts can be found in [14] and in [8].

Proposition 1. *An irreducible sofic shifts is of finite type if and only its syntactic graph is reduced to one node of rank 1 representing the trivial group.*

Another interesting class of irreducible sofic shifts can be characterized with the syntactic invariant. It is the class of aperiodic sofic shifts [1].

Let $x \in X$, we denote by $\text{period}(x)$ the least positive integer n such that $\sigma^n(x) = x$ if such an integer exists. It is equal to ∞ otherwise.

Let X, Y be two subshifts and let $\phi : X \to Y$ be a block map. The map is said *aperiodic* if $\text{period}(x) = \text{period}(\phi(x))$ for any $x \in X$. Roughly speaking, such a factor map ϕ does not make periods decrease.

A sofic shift X if *aperiodic* if it is the image of a shift of finite type by an aperiodic block map. A characterization of irreducible aperiodic sofic shifts is the following.

Proposition 2. *An irreducible sofic shift is aperiodic if and only if its syntactic graph contains only trivial groups.*

Schützenberger's characterization of aperiodic languages (see for instance [16, Theorem 2.1]) asserts that the set of blocks of an aperiodic sofic shift is a regular star free language.

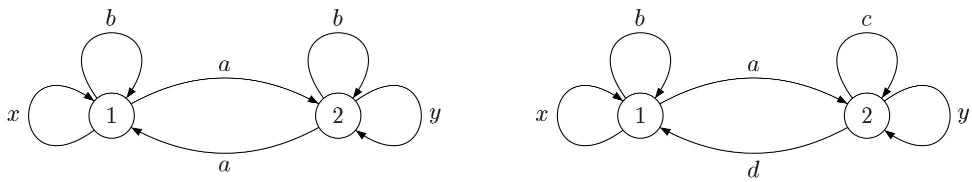


Fig. 7. Two sofic shifts X, Y which have the same zeta function $\frac{1}{1-4z+z^2}$ (see for instance [13, Theorem 6.4.8], or [2] for the computation of the zeta function of a sofic shift), and different syntactic invariants. Indeed the syntactic graph of X is (rank $2, \mathbb{Z}/2\mathbb{Z}$) $\to$ (rank $1, \mathbb{Z}/\mathbb{Z}$) while the syntactic graph of Y has only one node (rank $1, \mathbb{Z}/\mathbb{Z}$). Thus they are not conjugate. Notice that Y is a shift of finite type.

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